# CRACK GROWTH IN HYDRAULIC ROCK FRACTURING

Yu. A. Peslyak UDC 539.8

Various mining processes involve the injection of liquid under pressure into existing or newly-produced points; examples are hydraulic fracturing in oil, fracturing in coal seams, and oil displacement at elevated pressures [1, 2]. Studies have been made [3, 4] of vertical and horizontal crack growth in response to a noninfiltrating liquid. In those cases, the actual pressure distribution in a joint was replaced by the statically equivalent uniform pressure on part of the joint surface. Here we propose a treatment that handles such topics reasonably effectively and does not involve the assumption of uniform pressure distribution. A system of equations has been derived for a vertical symmetrical crack to define the Cauchy problem for the crack volume. The quasistatic equilibrium condition for the crack and the solution are very much simplified if a system of mobile elliptical coordinates related to the crack is used. An analogous approach has been used in examining the growth of a circular horizontal crack.

1. Consider a vertical crack disposed symmetrically with respect to a borehole.

As the crack propagation rate is small by comparison with the speed of sound, we can neglect dynamic effects and consider crack growth as a quasistatic process. Also, one can assume that the strain is described by the linear theory of elasticity, and that the liquid flow in the crack is laminar.

The projection of the crack on the horizontal xy plane at an instant t is represented by a section along the x axis from $-l(t)$ to $l(t)$; the state of strain in an unbounded rock body containing the crack may be determined by the method of [5] for planar deformation. As the situation is symmetrical, we have a condition for the bounded stresses at the ends of the crack:

$$\frac{1}{\pi}\int_{-l}^{l}\sqrt{\frac{l-x}{l+x}}\,p(x,t)\,dx-q_\infty l=0, \tag{1.1}$$

where p(x, t) is the pressure of the liquid in the crack, and $q_\infty=-\sigma_x(\infty)=-\sigma_y(\infty)$ is the lateral pressure in the undisrupted rock body.

The following is the normal component of displacement of points on the edges of the crack:

$$v_y^+(x,t)=-v_y^-(x,t)=\frac{4(1-\nu^2)}{E}\left[-q_\infty\sqrt{l^2-x^2}+\frac{1}{2\pi}\int_{-l}^{l}p(\xi,t)\ln\left|\frac{\sqrt{(l^2-x^2)(l^2-\xi^2)}+l^2-x\xi}{x-\xi}\right|d\xi\right], \tag{1.2}$$

where E and $\nu$ are the elastic modulus and the Poisson ratio for the rock.

The condition for flow continuity for an incompressible fluid in a crack is put in the form

$$\frac{dq}{\partial x}-Q\delta(x)=0,$$

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where q(x, t) is the volume flow rate of the liquid through the crack section, Q(t) is the flow rate of the liquid pumped into the crack, and $\delta(x)$ is a delta function.

We integrate the latter equation and use the symmetry to get

$$q(x)+\frac{Q}{2}[1-2\cdot 1(x)]=0,$$

where 1(x) is the unit Heaviside function (1(x) = 0 for x < 0 and 1(x) = 1 for x ≥ 0).

For planar laminar flow of the liquid we have

$$q=-h\frac{2d^3}{3\eta}\frac{\partial p}{\partial x},$$

where $2d = v_y^+ - v_y^- = 2v_y^+$ is the width of the crack, h is the thickness of the rock bed, and $\eta$ is the viscosity of the liquid.

At the ends of the growing crack $x = \pm l$, where $d \to 0$, the pressure gradient is $\partial p/\partial x \to \pm\infty$, and the flow rate is finite at $q(\pm l) = \pm Q/2$.

From the two latter equations we get the condition for continuity of the flow in the form

$$h\frac{2v^{+3}}{3\eta}\frac{\partial p}{\partial x}-\frac{Q}{2}[1-2\cdot 1(x)]=0. \tag{1.3}$$

The condition for conservation of the mass of liquid gives

$$2h\int_{-l}^{l}v_y^+(x,t)\,dx-\Omega=0, \tag{1.4}$$

where the volume of the crack is

$$\Omega=\Omega_0+\int_0^t Q\,dt. \tag{1.5}$$

For a given Q(t), Eqs. (1.1)-(1.4) constitute the system of equations for the unknown functions $l(t)$, $v_y^+(x, t)$ and p(x, t); we also have to specify the initial length and volume of the crack $l(0) = l_0$ and $\Omega(0) = \Omega_0$.

If the liquid flow rate is unknown, while the pressure at the borehole column $p_c$ is given, we add to these equations the following:

$$p(0,t)=p_c(t). \tag{1.6}$$

In that case, (1.1)-(1.6) defines the Cauchy problem for the crack volume $\Omega$ subject to the initial condition

$$\Omega(0)=\Omega_0. \tag{1.7}$$

In this formulation, the crack has a nonzero length at t = 0; the initial width and pressure distribution are defined by (1.1)-(1.4).

We introduce mobile elliptical coordinates $\rho$ and $\vartheta$ that are linked to the crack:

$$x=\frac{l}{2}\left(\rho+\frac{1}{\rho}\right)\cos\vartheta;\quad y=\frac{l}{2}\left(\rho-\frac{1}{\rho}\right)\sin\vartheta.$$

We have $\rho = 1$ at the edge of the crack, while the values $2n\pi \le \vartheta \le (2n+1)\pi$ correspond to $y = +0$ and $(2n+1)\pi \le \vartheta \le 2(n+1)\pi$ to $y = -0$ (n = 0, ± 1, ± 2, ...).

We put $p(\vartheta, t) = p[l(t)\cos\vartheta, t]$ and $v(\vartheta, t) = v_y^{\pm}[l(t)\cos(\vartheta, t)]$ to represent the above relationships as

$$\frac{1}{2\pi}\int_0^{2\pi} p(\vartheta, t)\,d\vartheta - q_\infty = 0; \tag{1.1a}$$

$$v(\vartheta, t) = -2l(t)\left[q_\infty \sin\vartheta + \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tg}\frac{\vartheta-\varphi}{2}\,d\varphi \int_0^{\varphi} p(\eta, t)\sin\eta\,d\eta\right]; \tag{1.2a}$$

$$\frac{2v^3(\vartheta, t)}{3l(t)\sin\vartheta}\frac{\partial p}{\partial\vartheta} - \frac{Q}{2h}r(2\vartheta) = 0; \tag{1.3a}$$

$$l(t)\int_0^{2\pi} v(\vartheta, t)\sin\vartheta\,d\vartheta - \frac{\Omega}{h} = 0; \tag{1.4a}$$

$$p\left(\frac{\pi}{2}, t\right) = p_c(t), \tag{1.6a}$$

where $r(\vartheta) = \operatorname{sign}(\sin\vartheta)$ is a Rademacher function, and introduce the following units for length $l_0$, stress $E/(1-\nu^2)$, and time $\eta[(1-\nu^2)/E]$, while for the dimensionless quantities we retain the symbols for the corresponding dimensional ones.

We represent the pressure in the form

$$p(\vartheta, t) = p_0(t) + \sum_{n=2,4,\ldots}^{\infty} p_n(t)\cos n\vartheta, \tag{1.8}$$

and get from (1.1a)-(1.4a) and (1.6a) that

$$p_0(t) = q_\infty; \tag{1.1b}$$

$$v(\vartheta, t) = l(t)\sum_{n=2}^{\infty} p_n(t)\left(\frac{\sin(n+1)\vartheta}{n+1} - \frac{\sin(n-1)\vartheta}{n-1}\right); \tag{1.2b}$$

$$\frac{4l^2(t)}{\sin\vartheta}\sum_{n=2}^{\infty} np_n\sin n\vartheta\left[\sum_{n=2}^{\infty} p_n\left(\frac{\sin(n+1)\vartheta}{n+1} - \frac{\sin(n-1)\vartheta}{n-1}\right)\right]^3 + \frac{Q}{h}r(2\vartheta) = 0, \tag{1.3b}$$

$$\left(r(\vartheta) = \frac{4}{\pi}\sum_{n=1,3,\ldots}^{\infty}\frac{\sin n\vartheta}{n}\right);$$

$$\pi l^2 p_2(t) + \frac{\Omega}{h} = 0; \tag{1.4b}$$

$$q_\infty + \sum_{n=2}^{\infty}(-1)^{\frac{n}{2}} p_n(t) = p_c(t). \tag{1.6b}$$

We multiply (1.3b) by $\sin n\vartheta$ and integrate from 0 to $2\pi$ and then put $n = 2, 4, 6, \ldots$, to get an infinite system of algebraic equations of fourth order for the coefficients $p_n(t)$; these equations go with (1.4b) to constitute a complete system of equations for all $p_n(t)$ and $l(t)$ for the given liquid flow rate. If the injection pressure is given, we use also (1.6b) and the initial condition of (1.7).

We restrict consideration to a single term in the sum of (1.8); then from (1.6b) we have $\Delta p_c(t) = p_c - q_\infty = -p_2(t)$, and from (1.3b), and (1.4b) we have

$$\frac{7}{27}\pi l^2\Delta p_c^4 - \frac{Q}{h} = 0; \tag{1.9}$$

$$\pi l^2\Delta p_c - \frac{\Omega}{h} = 0. \tag{1.10}$$

For $t = 0$ we have from (1.10) that

$$\Omega_0 = \pi h \Delta p_c(0). \tag{1.11}$$

For the given pressure we have from (1.9)-(1.11) that

$$\Omega = \pi h \Delta p_c(0) \exp \frac{7}{27} \int_0^t \Delta p_c^3(t)\, dt$$

and the other parameters of the process.

In particular, for a constant pressure in the column we have

$$Q = \frac{7}{27} \pi h \Delta p_c^4 \exp \frac{7}{27} \Delta p_c^3 t; \quad \Omega = \pi h \Delta p_c \exp \frac{7}{27} \Delta p_c^3 t;$$
$$l = \exp \frac{7}{54} \Delta p_c^3 t; \quad 2d_c = \frac{8}{3} \Delta p_c \exp \frac{7}{54} \Delta p_c^3 t,$$

where $2d_c$ is the width of the crack at the column.

All the parameters increase exponentially with time, the exponent being proportional to the cube of the parameter in the column and inversely proportional to the liquid viscosity and to the square of the elastic modulus of the rock.

It can be shown that incorporation of the subsequent terms in (1.8) merely alters the numerical factor in the exponent somewhat.

For a given flow rate we have from (1.9) and (1.10) that

$$\Delta p_c(t) = \left(\frac{27}{7} \frac{Q}{\Omega}\right)^{\frac{1}{3}}; \quad l(t) = \left(\frac{7}{27\pi^3} \frac{\Omega^4}{Qh^3}\right)^{\frac{1}{6}},$$

where $\Omega$ is defined by (1.5).

From (1.11) we have

$$\Delta p_c(0) = \left[\frac{27}{7\pi h} Q(0)\right]^{\frac{1}{4}}.$$

For a steady flow rate with a developed crack, where $\Omega \gg \Omega_0$ and consequently $\Omega = Qt$, we have

$$\Delta p_c = \left(\frac{27}{7}\right)^{1/3} t^{-1/3}; \quad l = \left(\frac{7}{27\pi^3}\right)^{1/6} \left(\frac{Q}{h}\right)^{1/2} t^{2/3};$$
$$2d_c = \frac{8}{3} \left(\frac{27}{7\pi}\right)^{1/6} \left(\frac{Q}{h}\right)^{1/2} t^{1/3}.$$

If the flow rate is constant, the pressure at the column is not dependent on the flow rate and falls as $t^{-1/3}$ with the passage of time [3]. The size of the crack increases without limit. The length of the crack increases by a 1/6 power law as the elastic modulus increases and the viscosity of the liquid falls, other conditions being unchanged; the crack width decreases to the same extent.

2. A study has previously been made [6, 7] of the strain in an infinite elastic medium with a planar circular crack with axially symmetrical deformation: the condition for finite stress of [7] is put in the form

$$\int_0^1 \frac{p(R\rho, t)\rho\, d\rho}{\sqrt{1-\rho^2}} - q_{z\infty} = 0, \quad \left(\rho = \frac{r}{R}\right), \tag{2.1}$$

where $p(r, t)$ is the pressure in the crack at distance $r$ from the center, $q_{z\infty} = -\sigma_z(\infty)$ is the vertical pressure in the undisrupted rock, and $R(t)$ is the crack radius.

We have the following for the normal displacement of the edges of the crack:

$$v_z^+(\rho, t) = -v_z^-(\rho, t) = \frac{4(1-\nu^2)R(t)}{\pi E}\Big[-q_{z\infty}\sqrt{1-\rho^2} + \int_\rho^1 \frac{\mu d\mu}{\sqrt{\mu^2-\rho^2}} \int_0^1 \frac{p(R\mu\xi, t)\xi d\xi}{\sqrt{1-\xi^2}}\Big]. \tag{2.2}$$

By analogy with the case of a vertical crack, we get the following relations:

$$\frac{2v_z^{+3}}{3\eta}\frac{\partial p}{\partial r} + \frac{Q}{2\pi r} = 0; \tag{2.3}$$

$$4\pi R^2 \int_0^1 v_z^+ \rho d\rho - \Omega = 0; \tag{2.4}$$

$$p(0, t) = p_c(t). \tag{2.5}$$

For t = 0 we have $R(0)-R_0$ and

$$\Omega(0) = \Omega_0.$$

We introduce mobile elliptical coordinates s and $\mu$ related to the cylindrical coordinates r and z by $r = R\sqrt{1+\mu^2}\sqrt{1-\mu^2}$, $z = Rs\mu$; at the surface of the crack we have $s = 0$, $r = R\sqrt{1-\mu^2}$, and the values $0 \le \mu \le 1$ correspond to $z = +0$, and $-1 \le \mu \le 0$ to $z = -0$.

We have the relationships $p(\mu, t) = p[R(t)[\sqrt{1-\mu^2}, t]$ and

$$v(\mu, t) = v_z^\pm\left[R(t)\sqrt{1-\mu^2}, t\right],$$

which allow (2.1)-(2.5) to be put as

$$-\frac{1}{2}\int_{-1}^{1} p(\mu, t)\,d\mu - q_{z\infty} = 0; \tag{2.1a}$$

$$v(\mu, t) = \frac{4}{\pi}R(t)\Big[-q_{z\infty}\mu + \int_0^\mu \frac{\eta d\eta}{\sqrt{1-\eta^2}\sqrt{\mu^2-\eta^2}} \int_\eta^1 \frac{p(\xi, t)\xi d\xi}{\sqrt{\xi^2-\eta^2}}\Big]; \tag{2.2a}$$

$$\frac{4\pi v^3}{3}\frac{1-\mu^2}{\mu}\frac{\partial p}{\partial \mu} - Q\,\mathrm{sign}\,\mu = 0; \tag{2.3a}$$

$$2\pi R^2(t)\int_{-1}^{1} v(\mu, t)\mu d\mu - \Omega = 0; \tag{2.4a}$$

$$p(1, t) = p_c(t), \tag{2.5a}$$

where, as in the case of a vertical crack, we have used dimensionless quantities. The unit of length is here $R_0$ instead of $l_0$.

The pressure of the liquid is put as

$$p(\mu, t) = p_c(t) + \sum_{n=2,4,\ldots}^{\infty} p_n(t)P_n(\mu), \tag{2.6}$$

where $P_n(\mu)$ are Legendre polynomials.

From (2.1a) we have

$$p_c(t) = q_{z\infty}. \tag{2.1b}$$

We restrict ourselves to a single term in sum of (2.6): then (2.1b) and (2.5a) give $\Delta p_c(t) = p_c(t) - q_{z\infty} = p_2(t)$, and from (2.2a)-(2.4a) we get

$$v(\mu, t) = \frac{8}{3\pi} R(t) \Delta p_c(t) \mu^3; \tag{2.2b}$$

$$\frac{2^{11}}{3^3\pi^3} R^3 \Delta p_c^4 (1-\mu^2) \mu^9 - Q \operatorname{sign} \mu = 0, \tag{2.3b}$$

$$\left(\operatorname{sign} \mu = \frac{3}{2} P_1(\mu) - \frac{7}{8} P_3(\mu) + \ldots\right);$$

$$\beta R^3 \Delta p_c - \Omega = 0, \quad \left(\beta = \frac{2^5}{3 \cdot 5} \approx 2.13\right). \tag{2.4b}$$

We multiply (2.3b) by $P_1(\mu) = \mu$ and integrate from $-1$ to $+1$ to get

$$\alpha R^3 \Delta p_c^4 - Q = 0, \quad \left(\alpha = \frac{2^{13}}{9 \cdot 11 \cdot 13\pi^3} \approx 0.208\right). \tag{2.7}$$

If the injection pressure at the column is constant, we have from (2.4b) and (2.7) that

$$Q = \alpha \Delta p_c^4 \exp \frac{\alpha}{\beta} \Delta p_c^3 t; \quad \Omega = \beta \Delta p_c \exp \frac{\alpha}{\beta} \Delta p_c^3 t;$$

$$R = \exp \frac{\alpha}{3\beta} \Delta p_c^3 t; \quad 2d_c = \frac{16}{3\pi} \Delta p_c \exp \frac{\alpha}{3\beta} \Delta p_c^3 t.$$

The growth of a horizontal circular crack is of the same type as that of a vertical crack.

If the flow rate is constant, we have for a developed crack that

$$\Delta p_c = \left(\frac{\beta}{\alpha}\right)^{1/3} t^{-1/3}; \quad R = \left(\frac{\alpha}{\beta^4}\right)^{1/9} Q^{1/3} t^{4/9};$$

$$2d_c = \frac{16}{3\pi} (\alpha^2 \beta)^{-1/9} Q^{1/3} t^{1/9}.$$

By analogy with the case of a vertical crack, the pressure at the column is independent of the flow rate and has the $t^{-1/3}$ time dependence; the crack radius increases in accordance with a 1/9 law as the elastic modulus of the rock increases and as the viscosity of the liquid falls; the crack width decreases as a 2/9 power.

3. We estimate the characteristic parameters of crack growth for $E = 10^5$ kgf/cm², $\nu = 0.2$, and $\eta = 1$ cP; we assume that initially the crack grows at a constant pressure $\Delta p_c = 10$ kgf/cm² up to some instant $t_1$, where the maximum output of the pump of 6 m³/min is reached. Then the crack receives 30 m³ of liquid for a period of 5 min at a constant flow rate Q of 6 m³/min.

For the case of the vertical crack we assume h = 10 m, $l_0 = 10$ cm; then we find $t_1 = 3.1$ sec, and at the end of the process we have $l = 297$ m, $2d_c = 0.86$ cm, $\Delta p_c = 1.12$ kgf/cm².

In the case of a horizontal crack with $R_0 = 10$ cm we have $t_1 = 15.3$ sec, and at the end of the pumping we have R = 101 m, $2d_c = 0.251$ cm, $\Delta p_c = 1.56$ kgf/cm².

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